

ON THE MAXIMAL DISPLACEMENT OF A CRITICAL BRANCHING RANDOM WALK

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ABSTRACT. We consider a branching random walk initiated by a single particle at location 0 in which particles alternately reproduce according to the law of a Galton-Watson process and disperse according to the law of a driftless random walk on the integers. When the offspring distribution has mean 1 the branching process is critical, and therefore dies out with probability 1. We prove that if the particle jump distribution has mean zero, positive finite variance η^2 , and finite $4 + \varepsilon$ moment, and if the offspring distribution has positive variance σ^2 and finite third moment then the distribution of the rightmost position M reached by a particle of the branching random walk satisfies $P\{M \geq x\} \sim 6\eta^2/(\sigma^2 x^2)$ as $x \rightarrow \infty$. We also prove a conditional limit theorem for the distribution of the rightmost particle location at time n given that the process survives for n generations.

1. INTRODUCTION

It has been known since the work of McKean [17] that the time evolution of a one-dimensional branching Brownian motion is intimately connected with the behavior of solutions of the Fisher-KPP equation

$$(1) \quad \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + f(u).$$

McKean observed that the cumulative distribution function of the position R_t of the rightmost particle at time t in a one-dimensional branching Brownian motion obeys equation (1). In the simplest case, where particles move independently along brownian trajectories and undergo simple binary fission following independent, exponentially distributed gestation times, the function f is given by $f(u) = u^2 - u$, the case originally studied by Fisher [12]; more general branching mechanisms lead to more general functional forms. In general, when the underlying branching process is *supercritical*, the solution of (1) with Heaviside initial data approaches a traveling wave with a positive asymptotic velocity, and thus, in particular, the distribution of R_t , when centered at its median m_t , converges weakly to the distribution

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described by the wave. There is now a considerable literature surrounding problems connected with this phenomenon: see, e.g., [8] for the precise asymptotic behavior of the function $t \mapsto m_t$; [15] for a proof that the traveling wave is a mixture of extreme value distributions; and [1] and [2] for proofs that the distribution of the entire point process of particle locations, when centered at m_t , converges in law as $t \rightarrow \infty$. Generalizations of some of these results to supercritical branching random walks are given in [4] and [7].

When the branching mechanism is *critical* (that is, when the mean number of offspring particles at a particle death is 1), the nature of the process R_t is entirely different, because in this case the process must ultimately die out, with probability one. Hence, it is more natural in the critical case to ask about the distribution of

$$(2) \quad M := \max_{t < \infty} R_t,$$

the rightmost point ever reached by a particle of the branching process. Interest in the distribution of M stems in part from its relevance to evolutionary biology, where critical branching Brownian motion has been used as a model for the spatial diffusion of alleles with no selective advantage or disadvantage: see, for instance, [9], [13], and the references therein. For critical (or subcritical) branching Brownian motion the distribution function $w(x) = P\{M \leq x\}$ of M satisfies the ordinary differential equation

$$(3) \quad \frac{1}{2}w''(x) = -\psi(w(x)),$$

where $\psi(w)$ is the probability generating function of the offspring distribution of the branching process. This differential equation is similar to that satisfied by the traveling wave(s) for the Fisher-KPP equation, but differs in the nonlinear term $\psi(w)$; this leads to solutions of a very different character, reflecting the qualitative differences between critical and supercritical branching. Using the equation (3), Fleischman and Sawyer [13] proved that if the offspring distribution has mean 1, positive variance, and finite third moment then

$$(4) \quad P\{M \geq x\} \sim \frac{6}{\sigma^2 x^2} \quad \text{as } x \rightarrow \infty.$$

(In the case of “double-or-nothing” branching, where particles produce either 0 or 2 offspring with probability 1/2, the solution to the differential equation with the appropriate boundary conditions has the closed form $1 - w(x) = 6/(1 + x)^2$.)

Our primary objective in this paper is to show that under suitable hypotheses the asymptotic formula (4) holds generally for *critical, driftless branching random walks*. For ease of exposition we will limit our study to discrete-time branching random walks on the integer lattice \mathbb{Z} in which the reproduction and dispersal mechanisms are independent, with dispersal *preceding* reproduction. Thus, in each generation, particles

- (A) jump (independently of one another) to new sites, with jumps following a *mean-zero, finite variance*, jump distribution $F_{RW} := \{a_x\}_{x \in \mathbb{Z}}$;
- (B) then reproduce as in a simple Galton-Watson process, according to a fixed offspring distribution $F_{GW} := \{p_k\}_{k \geq 0}$ with mean 1 and finite variance.

We will generally assume that the branching random walk is initiated by a single particle located at the origin $0 \in \mathbb{Z}$. For a formal construction of a branching random walk following rules (A)–(B), see, for instance, [14]. The locations of the particles in generation n will be denoted by $X_{n,i}$, where $i \leq N_n$ and N_n is the total number of particles in the n th generation. Our interest is in the distribution of the *maximal displacement*

$$(5) \quad M = \max_{n \geq 0} M_n \quad \text{where} \quad M_n = \max_{i=1,2,\dots,N_n} X_{n,i}.$$

Remark 1. Observe that only the particle locations at the end of each reproduction step are taken into account in the definition of M : thus, for instance, if the initial particle at $X_{0,1} = 0$ were to jump to site $x = 1$ and then produce no offspring, the maximal displacement would be $M = 0$, not $M = 1$.

Many authors (e.g., [14], [6]) consider branching random walks in which the order of the reproduction and jump steps is opposite to that specified above. For our purposes it is more convenient to have the jump step precede the reproduction step, as this leads to more compact formulations of the main result and the nonlinear convolution equation (11) below. It should be obvious that analogous results can be deduced for branching random walks of the type considered in [14], [6], by conditioning on the first jump step. See Remark 2 in sec. 2.1 below.

Theorem 1. *Assume that the step distribution $F_{RW} = \{a_k\}_{k \in \mathbb{Z}}$ has mean 0, positive variance η^2 , and finite r -th moment for some $r > 4$. Assume also that the offspring distribution $F_{GW} = \{p_k\}_{k \geq 0}$ has mean 1, positive variance σ^2 , and finite third moment. Then*

$$(6) \quad P\{M \geq x\} \sim \frac{6\eta^2}{\sigma^2 x^2} \quad \text{as } x \rightarrow \infty.$$

This will be proved in section 2. The result can also be reformulated as a statement about the distribution of the maximal displacement of a branching random walk initiated by a large number n of particles at the origin. Such a branching random walk is just the superposition (sum) of n independent copies of the branching random walk in Theorem 1, so the event that its maximal displacement is $\leq \sqrt{n}x$ is the intersection of the events that each of the n constituent branching random walks has maximal displacement $\leq \sqrt{n}x$. For fixed $x > 0$ the target points $\sqrt{n}x \rightarrow \infty$ as $n \rightarrow \infty$, so the asymptotic formula (6) yields the following corollary.

Corollary 2. *Let M^n be the maximal displacement of a branching random walk initiated by n particles at the origin at time 0. Then for any $x > 0$,*

$$(7) \quad \lim_{n \rightarrow \infty} P\{M^n \geq \sqrt{nx}\} = 1 - \exp\{-C/x^2\} \quad \text{where} \quad C = \frac{6\eta^2}{\sigma^2}.$$

Theorem 1 is closely related to the main results of Kesten [14], who considers critical branching random walk conditioned to survive for a large number of generations. (X. Zheng [19] has also considered the case where the random walk is assumed to have a “small” drift, but this leads to entirely different asymptotics.) Kesten shows that under the same hypotheses as in Theorem 1, for any fixed $\beta > 0$, given that the branching process survives for βn generations, the conditional distribution of $\max_{k \leq n} M_k / \sqrt{n}$ converges as $n \rightarrow \infty$. He remarks on the result of Sawyer and Fleischman:

We have not proved [equation (6)] in our setting. It is not clear at the moment how the methods of Sawyer and Fleischman, which rely on differential equations, can be carried over to the discrete setting; differential equations will have to be replaced by recurrence relations.

Our main technical innovation will be to show how to relate these “recurrence relations” to the differential equation (3). This will be accomplished by exploiting Feynman-Kac formulas. Our approach applies also to time-dependent Feynman-Kac problems, and leads in particular to information about the distribution of the random variable M_n . As an illustration, we will prove in section 3 the following conditional limit theorem.

Theorem 3. *Under the hypotheses of Theorem 1, the conditional distribution of M_n / \sqrt{n} , given that the branching process survives for n generations, converges weakly as $n \rightarrow \infty$ to a nontrivial limit distribution G that depends only on the variances σ^2 and η^2 of the offspring and step distributions.*

The scaling in this theorem is the same as that in the Dawson-Watanabe theorem (see, for instance, [10], ch. 1), which can be stated as follows. Suppose that n independent copies of the branching random walk are initiated at the origin $0 \in \mathbb{Z}$. If particles are assigned mass $1/n$, and if time and space are re-scaled by factors $1/n$ and $1/\sqrt{n}$, respectively, then the corresponding measure-valued processes $BRW_n(t)$ converge weakly to *super-Brownian motion* X_t . A similar theorem holds for the measure-valued processes $BRW_n^*(t)$ attached to branching random walk initiated by a single particle at the origin, but conditioned to survive for n generations: under the same re-scaling of mass, time, and space as in the Dawson-Watanabe theorem, the measure-valued processes $BRW_n^*(t)$ converge weakly as $n \rightarrow \infty$ to a measure-valued process Y_t . The law of this process Y_t is related to that of the super-Brownian

motion by the *Poisson clustering representation*: if N is a Poisson random variable with mean 1 and Y_t^1, Y_t^2, \dots are independent copies of Y_t then $\sum_{i=1}^N Y_t^i$ is a super-Brownian motion.

The weak convergence of the measure-valued processes $BRW_n^*(t)$ by itself does not imply Theorem 3, because the location M_n/\sqrt{n} is not a continuous function (relative to the weak topology on measures) of $BRW_n^*(1)$. (Lalley[16] shows that in dimension 1 rescaled branching random walks converge to super-Brownian motion in a stronger topology than the weak topology implicit in the Dawson-Watanabe theorem. However, even this topology is too weak to make the normalized rightmost particle location a continuous functional.) Nevertheless, it is natural to wonder whether how the limit distribution G of Theorem 3 is related to the limiting measure-valued process Y_t . The proof of Theorem 3 will establish that G is the distribution of the rightmost support point of the random measure Y_1 .

Corollary 4. *Under the hypotheses of Theorem 1 (in particular, under the assumption that the step distribution F_{RW} has finite r th moment for some $r > 4$),*

$$(8) \quad G(x) = P\{Y_1[x, \infty) = 0\}.$$

Are $4+\varepsilon$ moments on the step distribution really necessary for the validity of our theorems (and Kesten's)? The following simple heuristic argument (which with a bit of work can be made rigorous) shows that $4-\varepsilon$ moments are not enough. Consider, for instance, the case where F_{RW} is the symmetric distribution on the nonzero integers with discrete density

$$f_{RW}(x) = \frac{1}{2\zeta(5-\varepsilon)|x|^{5-\varepsilon}},$$

which has infinite 4th moment. Conditional on the event that the branching random walk survives for at least n generations it will produce on the order of n^2 particles. Each of these has conditional probability $\sim C/n^{2-\alpha}$ of placing an offspring at distance $n^{(1+\delta)/2}$ to the right, where $\alpha = \varepsilon/2 + \varepsilon\delta - 4\delta$. Consequently, if $\alpha > 0$ then for large n the probability that all n^2 particles are located in an interval $[-A\sqrt{n}, A\sqrt{n}]$ is vanishingly small.

2. MAXIMAL DISPLACEMENT: PROOF OF THEOREM 1

2.1. A Nonlinear Convolution Equation. For the remainder of the paper we shall assume that the offspring distribution $F_{GW} = \{p_k\}_{k \geq 0}$ and the jump distribution $F_{RW} = \{a_x\}_{x \in \mathbb{Z}}$ satisfy the hypotheses of Theorem 1: in particular, F_{GW} has mean 1, positive variance σ^2 , and finite third moment, and F_{RW} has mean 0, positive variance η^2 , and finite $4+\varepsilon$ moment. The maximal displacement M of the branching random walk is defined by equation (5), and its (tail) distribution function will be denoted by

$$(9) \quad u(x) = P\{M \geq x\}.$$

Clearly, $u(x) = 1$ for all $x \leq 0$, and $\lim_{x \rightarrow \infty} u(x) = 0$.

We begin by showing that u satisfies a nonlinear convolution equation analogous to the Fleischman-Sawyer equation (3). This is obtained in the conventional manner, by conditioning on the first generation of the branching process. Each particle of the first generation will give rise to its own descendant branching random walk, independent of its siblings (conditional on their locations), and in order that $M \leq x$ it must be the case that the maximal displacements of all of the descendant BRWs, adjusted by their starting points, must be $\leq x$. This implies that for all $x \geq 1$,

$$(10) \quad 1 - u(x) = \sum_{y \in \mathbb{Z}} a_y \sum_{k=1}^{\infty} p_k (1 - u(x - y))^k.$$

(Note that this is consistent with our convention regarding the sequencing of the dispersal and reproduction steps – see Remark 1 above.) Rewriting this equation in terms of u leads immediately to the following proposition.

Proposition 5. *$u(x)$ satisfies the nonlinear convolution equation*

$$(11) \quad u(x) = \sum_{k \in \mathbb{Z}} a_k Q(u(x - k)),$$

where $1 - Q(1 - s)$ is the probability generating function of the offspring distribution F_{GW} , that is,

$$(12) \quad Q(s) = 1 - \sum_{i=0}^{\infty} p_i (1 - s)^i, \quad \text{for } 0 \leq s \leq 1.$$

Remark 2. If the branching random walk used the alternative rule discussed in Remark 1 (that is, particles first reproduce and then disperse) to construct the branching random walk, then equation (11) would change as follows. Writing \tilde{M} for the maximal displacement of this branching random walk, and $\tilde{u}(x) = P\{\tilde{M} \geq x\}$, we would have

$$\tilde{u}(x) = Q\left(\sum_{k \in \mathbb{Z}} a_k \tilde{u}(x - k)\right).$$

Comparing this with equation (11), we see that

$$\tilde{u}(x) = Q(u(x)).$$

Since the Taylor expansion of Q is $Q(s) = s - \sigma^2 s^2 / 2 + O(s^3)$, it follows that $\tilde{u}(x)$ and $u(x)$ go to 0 as $x \rightarrow \infty$ at the same rate.

2.2. A Discrete Feynman-Kac Formula. Our goal now is to analyze the asymptotic behavior of solutions to the nonlinear convolution equation (11) as $x \rightarrow \infty$. To accomplish this, we will show that solutions of (11) can be represented by formulas of “Feynman-Kac” type. Henceforth, we shall

denote by W_n the random walk on \mathbb{Z} whose step distribution is the reflection of the step distribution F_{RW} in the underlying BRW, that is,

$$(13) \quad P(W_{n+1} - W_n = y \mid W_n, W_{n-1}, \dots) = a_{-y}.$$

We shall use superscripts P^x and E^x to denote the initial point $W_0 = x$ of the random walk W_n .

Define

$$(14) \quad \begin{aligned} h(s) &= s - Q(s) = \sigma^2 s^2 / 2 + O(s^3) \quad \text{and} \\ H(s) &= h(s)/s = \sigma^2 s / 2 + O(s^2). \end{aligned}$$

It is easily checked that $H(s)$ is increasing for $s \in [0, 1]$, and satisfies $H(0) = 0$ and $H(1) = p_0$.

Proposition 6. *Under P^x , the process*

$$(15) \quad Y_n = \left(\prod_{j=1}^n (1 - H(u(W_j))) \right) \cdot u(W_n)$$

is a bounded martingale with respect to the natural filtration generated by the random walk $\{W_n\}$. Here we use the convention that the empty product $\prod_{j=1}^0$ is equal to 1.

Proof. The random variable $\{Y_n\}$ is clearly bounded: $0 \leq Y_n \leq 1$. To prove that the sequence is a martingale we appeal to the fundamental convolution equation (11), which can be rewritten in terms of the function h as

$$(16) \quad \left(\sum_{k \in \mathbb{Z}} a_k u(x - k) \right) - u(x) = \sum_{k \in \mathbb{Z}} a_k h(u(x - k)).$$

Using this, we compute

$$\begin{aligned} & E^x(Y_{n+1} \mid \{W_j\}_{j=1}^n) \\ &= \left(\prod_{j=1}^n (1 - H(u(W_j))) \right) \cdot E^x((1 - H(u(W_{n+1}))) \cdot u(W_{n+1}) \mid \{W_j\}_{j=1}^n) \\ &= \left(\prod_{j=1}^n (1 - H(u(W_j))) \right) \cdot E^x((u(W_{n+1}) - h(u(W_{n+1}))) \mid \{W_j\}_{j=1}^n) \\ &= \left(\prod_{j=1}^n (1 - H(u(W_j))) \right) \cdot \left(\sum_{k \in \mathbb{Z}} a_k u(W_n - k) - \sum_{k \in \mathbb{Z}} a_k h(u(W_n - k)) \right) \\ &= \left(\prod_{j=1}^n (1 - H(u(W_j))) \right) \cdot u(W_n) = Y_n, \end{aligned}$$

where the second to last equality uses the fundamental equation (16). \square

Corollary 7. *For each $y \in \mathbb{Z}$, define*

$$(17) \quad \tau_y = \min\{n \geq 0 \mid W_n \leq y\}.$$

Then for all $x, y \in \mathbb{Z}$,

$$(18) \quad u(x) = E^x \left(\prod_{j=1}^{\tau_y} (1 - H(u(W_j))) \right) u(W_{\tau_y}).$$

Proof. Since the random walk W_n is driftless it must be recurrent, and hence τ_y is finite. Since the martingale Y_n of Proposition 6 is bounded, Doob's optional sampling identity applies, yielding (18). Note that if $y \leq 0$ then $u(W_{\tau_y}) = 1$, since $W_{\tau_y} \leq 0$. \square

2.3. Scaling Limits. Using the Feynman-Kac representation (18) we will show that the function u , properly re-normalized, converges to a function that satisfies the Fleischman-Sawyer equation (3). Because we do not know *a priori* that the function u has a proper scaling limit we must work with subsequential limits. However, since u is monotone and satisfies $0 < u \leq 1$ it follows that for any $y \geq 0$ there exist sequences $x_k \rightarrow \infty$ such that

$$(19) \quad \phi(y) := \lim_{k \rightarrow \infty} \frac{u(x_k + y/\sqrt{u(x_k)})}{u(x_k)}$$

exists. Clearly, the limit must satisfy $0 \leq \phi(y) \leq 1$, and if $y = 0$ the limit is $\phi(0) = 1$. By Cantor's diagonalization argument, any such sequence x_k must have a subsequence, which we also denote by x_k , such that the convergence (19) holds for all *rational* $y \geq 0$.

Proposition 8. *For any sequence $x_k \rightarrow \infty$ such that (19) holds for all rational $y \geq 0$, the limit function $\phi(y)$ extends to a continuous, non-increasing, positive function of $y \in [0, \infty)$. Hence, the convergence (19) holds uniformly for y in any compact interval $[0, A]$.*

Proof. Fix $0 \leq y_1 < y_2$, both rational, and for ease of notation write $z_i = y_i/\sqrt{u(x)}$ for $i = 1, 2$. Fix a sequence $x_k \rightarrow \infty$ along which (19) holds for all rational y . To avoid a proliferation of subscripts, we shall write $\lim_{x \rightarrow \infty}$ to mean convergence along the subsequence x_k . By Proposition 6 and Doob's optional sampling theorem,

$$\begin{aligned} \phi(y_2) &= \lim_{x \rightarrow \infty} \frac{u(x + z_2)}{u(x)} \\ &= \lim_{x \rightarrow \infty} E^{x+z_2} \left(\frac{u(W_{\tau(x+z_1)})}{u(x)} \prod_{j=1}^{\tau(x+z_1)} (1 - H(u(W_j))) \right), \end{aligned}$$

where $\tau_z = \tau(z) = \min\{j \geq 0 \mid W_j \leq z\}$. Using the expansion $H(u) \sim \sigma^2 u/2$ as $u \rightarrow 0$ we obtain

$$\begin{aligned} \phi(y_2) &= \lim_{x \rightarrow \infty} E^{x+z_2-z_1} \left(\prod_{j=1}^{\tau(x)} (1 - H(u(W_j + z_1))) \frac{u(W_{\tau(x)} + z_1)}{u(x)} \right) \\ &= \lim_{x \rightarrow \infty} E^{x+z_2-z_1} \left(\exp \left\{ -\frac{\sigma^2}{2} \sum_{j=1}^{\tau(x)} u(W_j + z_1) \right\} \frac{u(W_{\tau(x)} + z_1)}{u(x)} \right) \\ &\geq \lim_{x \rightarrow \infty} E^{x+z_2-z_1} \left(\exp \left\{ -\frac{\sigma^2}{2} \tau(x) u(x) \right\} \frac{u(x + z_1)}{u(x)} \right). \end{aligned}$$

The last inequality holds because $W_j > x$ for all $j < \tau_x$ and $W_{\tau_x} \leq x$.

By the invariance principle (here we use the assumption that the step distribution of the random walk W_n has mean 0 and finite variance), as $x \rightarrow \infty$,

$$\begin{aligned} \mathcal{D}(\tau_x u(x) \mid W_0 = x + z_2 - z_1) &= \mathcal{D}(\tau_0 u(x) \mid W_0 = z_2 - z_1) \\ &\implies \mathcal{D}(\tau_0^{BM} \mid B_0 = y_2 - y_1) \end{aligned}$$

where B_t is a standard Brownian motion started at $y_2 - y_1$ and τ_0^{BM} is the first hitting time of 0 by B_t . Hence, for any $\epsilon > 0$, when $y_2 - y_1$ is sufficiently small and x is sufficiently large, $\exp\{-\sigma^2 \tau_x u(x)/2\} \geq 1 - \epsilon$ with probability $1 - \epsilon$. Consequently,

$$\begin{aligned} &\lim_{x \rightarrow \infty} E^{x+z_2-z_1} \left(\exp\left(-\frac{\sigma^2}{2} \tau_x u(x)\right) \frac{u(x + z_1)}{u(x)} \right) \\ &\geq (1 - \epsilon)^2 \lim_{x \rightarrow \infty} \frac{u(x + z_1)}{u(x)} \\ &= (1 - \epsilon)^2 \phi(y_1). \end{aligned}$$

This proves that $\forall \epsilon > 0$, if $z_2 - z_1$ is sufficiently close to zero then $\phi(y_1) \geq \phi(y_2) \geq (1 - \epsilon)^2 \phi(y_1)$. Therefore, ϕ is continuous. \square

Our aim now is to show that there is only one possible subsequential limit function ϕ , and that it satisfies the Fleischman-Sawyer differential equation. To accomplish this we will use the discrete Feynman-Kac formula (18) and the invariance principle to show that any subsequential limit ϕ satisfies the following Feynman-Kac formula.

Proposition 9. *Assume that the step distribution $\{a_k\}_{k \in \mathbb{Z}}$ of the random walk $\{W_n\}$ has finite r -th moment for some $r > 4$. Then any subsequential limit $\phi(y)$ specified by (19) satisfies*

$$(20) \quad \phi(y) = E^{y/\eta} \exp\left(-\frac{\sigma^2}{2} \int_0^{\tau_0^{BM}} \phi(\eta B_t) dt\right),$$

where under $P^{y/\eta}$ the process B_t is a standard Brownian motion started at $B_0 = y/\eta$ and τ_0^{BM} is the first hitting time of 0 by B_t .

The need for finite $4 + \varepsilon$ moment stems from the fact that in general the random walk W_n will overshoot 0 at the first passage time $\tau(0)$. The renewal theorem (cf. [11] or [18]) implies that as the initial point $W_0 = x \rightarrow \infty$, the distribution of the overshoot $W_{\tau(0)}$ converges weakly provided the step distribution of the random walk has mean zero and finite variance. The number of finite moments of the limiting overshoot distribution is determined by the number of moments of the step distribution, as follows.

Lemma 10. *If the step distribution of $\{W_n\}$ has finite r -th moment, then the limiting overshoot distribution has finite $(r - 2)$ -th moment.*

Proof. Consider the ladder variables

$$\begin{aligned} T_1 &= \min\{n > 0 \mid W_n < W_0\}, & Z_1 &= W_{T_1} - W_0, \\ T_2 &= \min\{n > T_1 \mid W_n < W_{T_1}\}, & Z_2 &= W_{T_2} - W_{T_1}, \end{aligned}$$

and so on. Clearly, the first passage time $\tau(0)$ must be one of the ladder times T_i . Moreover, the ladder steps $Z_{m+1} - Z_m$ are i.i.d., and by exercise 6, p. 232 of [18], the random variable Z_1 has finite absolute r th moment if the step distribution F_{RW} has finite absolute $(r + 1)$ th moment. The key observation is that for any $a \leq 0$ and any $x \geq 1$,

$$\begin{aligned} P^x(W_{\tau(0)} \leq a) &= \sum_{k=1}^x G(x; k) P^k\{Z_1 \leq a - k\} \\ &\leq \sum_{k=1}^x P^k\{Z_1 \leq a - k\} \\ &= \sum_{k=1}^x P^0\{Z_1 \leq a - k\} \end{aligned}$$

where $G(x; k)$ is the probability under P^x that the random walk W_n will visit the site k at one of the ladder times T_i . Thus, if S has the limiting overshoot distribution, then for any $a \leq 0$,

$$P(S \leq a) \leq \sum_{k=1}^{\infty} P(Z_1 \leq a - k).$$

This inequality together with the earlier observation about moments of the ladder variable Z_1 implies that if $E^0|W_1|^r < \infty$ then S has at least $r - 2$ moments. \square

Lemma 11. *If the step distribution F_{RW} satisfies the hypotheses of Theorem 1 then along any sequence $x = x_k \rightarrow \infty$ such that (19) holds uniformly on compact sets,*

$$(21) \quad \lim_{k \rightarrow \infty} E^{y/\eta} \sqrt{u(x_k)} \left(\frac{u(W_{\tau_0} + x_k)}{u(x_k)} \right) = 1.$$

Proof. As in the proof of Proposition 8 we will omit the subscript k on x_k and write $\lim_{x \rightarrow \infty}$ to mean convergence along the subsequence x_k . We also write $z = y / \sqrt{u(x)}$. By Proposition 8,

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow \infty} \frac{u(x+z)}{u(x)} = 1,$$

which implies that for any $\alpha \in (0, 1/2)$,

$$\lim_{x \rightarrow \infty} \frac{u(x + u(x)^{-\frac{1}{2} + \alpha})}{u(x)} = 1.$$

By the monotonicity of u ,

$$\begin{aligned} 1 &\leq E^z \left(\frac{u(W_{\tau_0} + x)}{u(x)} \right) \\ &= E^z \left(\frac{u(W_{\tau_0} + x)}{u(x)} \right) \mathbf{1}_A + E^z \left(\frac{u(W_{\tau_0} + x)}{u(x)} \right) \mathbf{1}_{A^c} \\ &= I + II, \end{aligned}$$

where

$$A = A(x) = \{W_{\tau(0)} \geq -u(x)^{-1/2+\alpha}\}.$$

By Chebyshev's inequality, for any $r > 2$,

$$P^x(A^c) \leq E|W_{\tau(0)}|^{r-2} u(x)^{(r-2)(\frac{1}{2}-\alpha)},$$

and by Lemma 10, if the step distribution F_{RW} has finite r th moment then $E|W_{\tau(0)}|^{r-2} < \infty$. Consequently,

$$II \leq C(u(x))^{(\frac{1}{2}-\alpha)(r-2)-1}.$$

Since by hypothesis the step distribution F_{RW} has finite r th moment for some $r > 4$, the constant $\alpha > 0$ can be chosen so that the exponent in the last displayed inequality is positive, and so it follows that quantity II converges to 0 as $x \rightarrow \infty$. On the other hand, $\lim_{x \rightarrow \infty} P^x(A) = 1$, and on the event A the integrand in quantity I is bounded by

$$\frac{u(x - u(x)^{-1/2+\alpha})}{u(x)} \rightarrow 1,$$

and so quantity I converges to 1 as $x \rightarrow \infty$. □

Proof of Proposition 9. Once again write $z = y/\sqrt{u(x)}$. According to Proposition 6 and the Optional Stopping Theorem,

$$\begin{aligned}
 & \frac{u(x + y/\sqrt{u(x)})}{u(x)} \\
 &= E^{x+z} \prod_{j=1}^{\tau_x} (1 - H(u(W_j))) \frac{u(W_{\tau_x})}{u(x)} \\
 &= E^z \prod_{j=1}^{\tau_0} (1 - H(u(W_j + x))) \frac{u(W_{\tau_0} + x)}{u(x)} \\
 (22) \quad &= E^z \exp\left\{\sum_{j=1}^{\tau_0} \log(1 - H(u(W_j + x)))\right\} \frac{u(W_{\tau_0} + x)}{u(x)} \\
 &= E^z \exp\left\{\sum_{j=1}^{\tau_0} \left(-\frac{\sigma^2}{2} u(W_j + x) + O(u(x)^2)\right)\right\} \frac{u(W_{\tau_0} + x)}{u(x)} \\
 &= E^z \exp\left\{-\frac{\sigma^2}{2} \sum_{j=1}^{\tau_0} u(W_j + x) + \tau_0 O(u(x)^2)\right\} \frac{u(W_{\tau_0} + x)}{u(x)}
 \end{aligned}$$

The error term $O(u(x)^2)$ is bounded in magnitude by $Cu(x)^2$ for some finite constant C not depending on x , by virtue of our standing hypothesis that the offspring distribution F_{GW} has finite third moment, which ensures that $H(u)$ has finite third derivative at $u = 0$.

The invariance principle implies that as $x \rightarrow \infty$ the distribution of the process $\sqrt{u(x)}W_{t/u(x)}/\eta$ under P^z converges weakly to that of a standard Brownian motion B_t started at $B_0 = y/\eta$. Consequently, the distribution of the renormalized first passage time $u(x)\tau_0$ converges weakly to that of τ_0^{BM} , and hence the error term $\tau_0 O(u(x)^2)$ converges in distribution to 0. Moreover, along any sequence $x = x_k \rightarrow \infty$ such that the convergence (19) holds uniformly for y in compact intervals,

$$\sum_{j=1}^{\tau_0} u(W_j + x) = u(x) \sum_{j=1}^{\tau_0} u(W_j + x)/u(x) \xrightarrow{\mathcal{D}} \int_0^{\tau_0^{BM}} \phi(\eta B_t) dt.$$

Therefore, by Lemma 11,

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} E^z \exp\left\{-\frac{\sigma^2}{2} \sum_{j=1}^{\tau_0} u(W_j + x_k) + \tau_0 O(u(x_k)^2)\right\} \frac{u(W_{\tau_0} + x_k)}{u(x_k)} \\
 &= E^{y/\eta} \exp\left\{-\frac{\sigma^2}{2} \int_0^{\tau_0^{BM}} \phi(\eta B_t) dt\right\}.
 \end{aligned}$$

This together with the convergence (19) and the chain of equalities (22) proves that ϕ must satisfy the Feynman-Kac formula (20). \square

Corollary 12. *Under the hypotheses of Proposition 9,*

$$(23) \quad \lim_{x \rightarrow \infty} \frac{u(x + y/\sqrt{u(x)})}{u(x)} = \left(\frac{\sigma y}{\sqrt{6}\eta} + 1 \right)^{-2} := \phi(y).$$

Proof. It suffices to prove that there is only one possible subsequential limit function (19), and that this limit is the solution of the differential equation

$$(24) \quad \phi''(y) = \sigma^2 \phi(y)^2 / \eta^2$$

that satisfies $\phi(0) = 1$ and $\lim_{y \rightarrow \infty} \phi(y) = 0$. This follows from the Feynman-Kac representation (20) of subsequential limits and Kac's theorem (cf., for instance, Theorem 3.7, ch. II of [5]), which implies that for any positive, bounded, continuous function $V : [0, \infty) \rightarrow \mathbb{R}$ the function

$$\psi(y) = E^{y/\eta} \exp \left\{ -\frac{\sigma^2}{2} \int_0^{\tau_0^{BM}} V(\eta B_t) dt \right\}$$

is the unique bounded solution of the differential equation $\psi'' = \sigma^2 V \psi / \eta^2$ satisfying $\psi(0) = 1$. \square

2.4. Proof of Theorem 1. To complete the proof of Theorem 1 we must show that

$$(25) \quad \lim_{x \rightarrow \infty} w(x) = 1/\beta^2 := 6\eta^2/\sigma^2 \quad \text{where} \quad w(x) := x^2 u(x).$$

This we will deduce from the asymptotic scaling law (23), which in terms of the function w may be rewritten as

$$(26) \quad \lim_{x \rightarrow \infty} \frac{(1 + \beta y)^2}{(1 + y/\sqrt{w(x)})^2} \cdot \frac{w(x(1 + y/\sqrt{w(x)}))}{w(x)} = 1.$$

Lemma 13. $\limsup_{x \rightarrow \infty} w(x) < \infty$.

Proof. First we show that $\liminf_{x \rightarrow \infty} w(x) < \infty$. If not, then for all sufficiently large k the function w would assume the value k for a last time, at $z_k := \sup\{x \mid w(x) \leq k\} < \infty$, and $w(z_k) = k$ by the continuity of w . Since $w(z) > k$ for all $z > z_k$, it would then follow that for any $y > 0$,

$$\frac{w(z_k(1 + y/\sqrt{k}))}{w(z_k)} \geq 1$$

But

$$\lim_{k \rightarrow \infty} \frac{(1 + \beta y)^2}{(1 + y/\sqrt{k})^2} = (1 + \beta y)^2 > 1,$$

so this would contradict the relation (26).

Next, suppose that $\limsup_{x \rightarrow \infty} w(x) = \infty$. Since $\liminf_{x \rightarrow \infty} w(x) < \infty$, by the preceding argument, for any sufficiently large A there exist sequences $x_k \geq \tilde{x}_k \rightarrow \infty$ such that $w(\tilde{x}_k) = A^2/\beta^2$ for all k but $\lim_{k \rightarrow \infty} w(x_k) = \infty$. Define y_k by

$$x_k = \tilde{x}_k(1 + \beta y_k/A);$$

then

$$\frac{(1 + \beta y_k)^2}{(1 + y_k/\sqrt{w(\tilde{x}_k)})^2} = \frac{(1 + \beta y_k)^2}{(1 + \beta y_k/A)^2} \geq 1.$$

But by construction,

$$\frac{w(\tilde{x}_k(1 + y_k/\sqrt{w(\tilde{x}_k)}))}{w(\tilde{x}_k)} = \frac{w(x_k)}{w(\tilde{x}_k)} \rightarrow \infty,$$

so again we have a contradiction to (26). \square

Proof of Theorem 1. We must establish that $\lim_{x \rightarrow \infty} w(x) = 1/\beta^2$. Let $x_k \rightarrow \infty$ be any sequence along which $w(x_k)$ converges, and denote $L = \lim_{k \rightarrow \infty} w(x_k)$. By Lemma 13, the limit point L must be finite. Fix $y > 0$, and choose $\tilde{x}_k = \tilde{x}_k < x_k$ such that

$$x_k = \tilde{x}_k(1 + y/\sqrt{w(\tilde{x}_k)}) = \tilde{x}_k + y/\sqrt{u(\tilde{x}_k)}.$$

Such points $\tilde{x}_k \geq 0$ must exist, provided $x_k \geq y$, because the right hand side increases continuously with \tilde{x}_k on the interval $[0, x_k]$. Moreover, for any fixed $y > 0$ the sequence $\tilde{x}_k(y) \rightarrow \infty$, since $x_k \rightarrow \infty$. By going to a subsequence of x_k if necessary we may arrange that the sequence $w(\tilde{x}_k)$ converges, with limit $K = K(y) < \infty$. The fundamental relation (26) (using $x = \tilde{x}_k$) now implies that

$$\frac{(1 + \beta y)^2}{(1 + y/\sqrt{K})^2} \cdot \frac{L}{K} = \frac{(\sqrt{L} + \beta \sqrt{L}y)^2}{(\sqrt{K} + y)^2} = 1,$$

and hence

$$(27) \quad \sqrt{K(y)} = \sqrt{L} + (\beta \sqrt{L} - 1)y.$$

By Lemma 13, the set of limit points $K(y)$ is bounded above. Since this must hold for every $y > 0$, it follows from (27) that $\beta^2 L \leq 1$. On the other hand, for every $y \in \mathbb{R}$ every limit point $K(y)$ must be nonnegative, so (27) implies that $\beta^2 L \geq 1$. \square

3. CONDITIONAL LIMIT THEOREM

3.1. Space-time Feynman-Kac formula. Assume throughout this section that M_n is the rightmost particle location in the n th generation of a branching random walk satisfying the hypotheses of Theorem 1. The *unconditional* distribution function of the random variable M_n satisfies a time-dependent

nonlinear convolution equation similar to the time-independent equation (11) satisfied by the distribution of the maximal displacement random variable M . Specifically, if

$$(28) \quad v_n(x) = v(n, x) := P\{M_n \geq x\},$$

then for every $n \geq 1$,

$$(29) \quad v_n(x) = \sum_{y \in \mathbb{Z}} a_k Q(v_{n-1}(x - y)),$$

where $1 - Q(1 - s)$ is the probability generating function of the offspring distribution (see equation (12)). The objective of this section is to analyze the asymptotic behavior of v , and in particular to show that $nv(n, [x\sqrt{n}])$ converges as $n \rightarrow \infty$, for any $x \in \mathbb{R}$, to a distribution function that depends only on the variances of the offspring and step distributions of the branching random walk. The strategy will once again be to represent the solution of (29) by a discrete Feynman-Kac formula, and then to show that after an appropriate rescaling the Feynman-Kac expectations converge to a Feynman-Kac expectations for Brownian motion. As in sec. 2, denote by W_n a random walk with step distribution (13), and indicate by P^x the law of the random walk with initial point $W_0 = x$. Then essentially the same arguments as in the time-independent case prove the following assertion.

Proposition 14. *For each n the process*

$$(30) \quad Z_k^{(n)} = v_{n-k}(W_k) \prod_{j=1}^{k-1} (1 - H(v_{n-j}(W_j)))$$

is a martingale under P^x . Consequently, for any $n \leq m$,

$$(31) \quad v_m(x) = E^x v_n(W_{m-n}) \prod_{j=1}^{m-j} (1 - H(v_{m-j}(W_j))).$$

3.2. Monotonicity, tightness, and scaling limits. For each $n \geq 0$ the function $v_n(x)$ is non-increasing in x , with limit 0 as $x \rightarrow \infty$ and limit $v_n(-\infty) = P\{\zeta \geq n\}$ as $x \rightarrow -\infty$, where ζ is the extinction time of the branching random walk. By Kolmogorov's theorem on the lifetime of a critical Galton-Watson process (see [3], ch. 1), as $n \rightarrow \infty$,

$$(32) \quad v_n(-\infty) = P\{\zeta \geq n\} \sim \frac{2}{\sigma^2 n}.$$

Hence, the functions $nv_n(x)$ are uniformly bounded.

Lemma 15. *Under the hypotheses of Theorem 1, the family of rescaled distribution functions $nv_n(x\sqrt{n})$ is tight, that is,*

$$(33) \quad \lim_{x \rightarrow \infty} \sup_{n \geq 1} nP\{M_n \geq x\sqrt{n}\} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \sup_{n \geq 1} nP\{M_n \leq x\sqrt{n}\} = 0.$$

Proof. The first of these follows directly from Theorem 1, because $M_n \leq M$. The second follows from Theorem 1 by reflection of the branching random walk in the origin. \square

Remark 3. The hypothesis that the step distribution F_{RW} of the branching random walk has finite $4 + \varepsilon$ moment is used here in an essential way. If F_{RW} has infinite $4 - \varepsilon$ moment for some $\varepsilon > 0$ then Lemma 15 need not be true: see the discussion at the end of sec. 2.

Lemma 16. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| \leq \delta\sqrt{n}$ and $n \leq m \leq n(1 + \delta)$ then*

$$(34) \quad \left| \frac{v_m(y)}{v_n(x)} - 1 \right| \leq \varepsilon.$$

Proof. This follows routinely from the Feynman-Kac formula (31) and Donsker's invariance principle, since $\sup_x v_n(x) \rightarrow 0$ as $n \rightarrow \infty$ and $H(u) \sim \sigma^2 u/2$ as $u \rightarrow 0$. \square

Corollary 17. *Any sequence of positive integers has a subsequence $n_k \rightarrow \infty$ along which the functions $(t, x) \mapsto nv_{[nt]}(x\sqrt{n})$ converge uniformly for $1 \leq t \leq A$ and $x \in [-\infty, \infty]$, for any $A < \infty$. The set of possible limit functions $\varphi(t, x)$ is compact in $C([1, A] \times \mathbb{R})$, and for each t the function $\varphi(t, x)$ is non-increasing in x , with*

$$(35) \quad \lim_{x \rightarrow \infty} \varphi(t, x) = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} t\varphi(t, x) = 2/\sigma^2.$$

Proof. All of the assertions except the limits (35) follow from Lemma 16. The limit relations (35) follow from the tightness of the family $nv_n(x\sqrt{n})$. \square

Corollary 18. *Let $\varphi(t, x)$ be any subsequential limit of the functions $(t, x) \mapsto nv_{[nt]}(x\sqrt{n})$, for $1 \leq t$ and $x \in [-\infty, \infty]$. Then φ satisfies the identity*

$$(36) \quad \varphi(t+1, x) = E^{x/\eta} \varphi(1, \eta B_t) \exp \left\{ -\frac{\sigma^2}{2} \int_0^t \varphi(t+1-s, \eta B_s) ds \right\},$$

where under P^y the process B_t is a standard Brownian motion started at y . Consequently, φ satisfies the partial differential equation

$$(37) \quad \frac{\partial \varphi}{\partial t} = \frac{\eta^2}{2} \frac{\partial^2 \varphi}{\partial x^2} - \sigma^2 \varphi^2 \quad \text{for } t > 1 \quad \text{and } x \in \mathbb{R}.$$

Proof. The integral representation (36) follows from the discrete Feynman-Kac formula (31) and the invariance principle by virtually the same argument as in the proof of Proposition 9. The differential equation (37) follows from the integral representation by Kac's theorem. \square

It should be noted that the only *a priori* bounds on the functions v_n needed to deduce the existence of subsequential limits $\varphi(t, x)$ are those in Lemmas 15–16 and Corollary 17. These use only the crude estimate $M_n \leq M := \max_{n \geq 1} M_n$ and the results concerning the tail behavior of the

distribution of M proved in section 2.2. To prove that there is only one possible subsequential limit function $\varphi(t, x)$ we will need the following stronger *a priori* bounds on the functions v_n .

Lemma 19.

$$(38) \quad \lim_{x \rightarrow \infty} \sup_{n \geq 1} nx^2 v_n(x \sqrt{n}) = 0 \quad \text{and}$$

$$(39) \quad \lim_{x \rightarrow -\infty} \inf_{n \geq 1} nx^2 (1 - v_n(x \sqrt{n})) = 0.$$

The proof is deferred to section 3.6 below.

3.3. Super-Brownian motion and solutions of (37). At first sight it might appear that Corollary 18 sheds no light at all on the question of uniqueness of scaling limits, because in general the solution to the partial differential equation (37) will depend on the initial condition $\varphi(1, x)$. However, we will show, using the *a priori* bounds in Lemma 19, that in fact Corollary 18 implies that there can be only one scaling limit, and thereby complete the proof of Theorem 3. The key is the fact that solutions of (37) determine – and are determined by – the law of super-Brownian motion, by the following *duality formula*. For ease of exposition, assume henceforth that $\eta^2 = 1$. (There is no loss of generality in this, because solutions of (37) can be rescaled.)

Proposition 20. Let $X_t^{\delta_x}$ be a super-Brownian motion with branching parameter σ^2 and initial mass distribution $X_0^{\delta_x} = \delta_x$, a unit point mass at location $x \in \mathbb{R}$. Let $\varphi(t+1, x)$ be the solution of the evolution equation (37) with initial condition $\varphi(1, x) = \varphi(x)$. Then

$$(40) \quad \varphi(t+1, x) = -\log E \exp\{-\langle X_t^{\delta_x}, \varphi \rangle\}.$$

Here $\langle \mu, \varphi \rangle$ denotes the integral of φ against the measure μ .

Proof. See, for instance, [10], Cor. 1.25. □

Exploitation of the duality formula (40) will require some elementary properties of super-Brownian motion. First, super-Brownian motion is equivariant under spatial translation: $X_t^{\delta_x}$ can be obtained from $X_t^{\delta_0}$ by translating each of the random measures $X_t^{\delta_0}$ by x to the right. Second, super-Brownian motion satisfies a *scaling law*: if \tilde{X}_t is a super-Brownian motion with initial mass distribution $\tilde{X}_0 = A\delta_{x/\sqrt{A}}$ then the rescaled measure-valued process X_t defined by

$$\langle X_t, f \rangle = \langle A^{-1} \tilde{X}_{At}, f(\sqrt{A} \cdot) \rangle$$

is a super-Brownian motion with initial mass distribution $X_0 = \delta_x$. Third, super-Brownian motion is *infinitely divisible*, in the following sense: the superposition of m independent super-Brownian motions with initial mass distributions ν_i is a super-Brownian motion with initial mass distribution

$\sum_{i=1}^m \nu_i$. This implies that the total mass $|X_t|$ at time t evolves as *Feller diffusion*, and so $P\{X_t \neq 0\} \sim C/t$ as $t \rightarrow \infty$ with $C = 2/\sigma^2$, where σ^2 is the branching parameter (cf. [?], sec. 2.1, Theorem 1 and Example (iii)). It follows, by (40), that any solution $\varphi(t+1, x)$ of (37) whose initial condition $\varphi(x) = \varphi(1, x)$ is non-increasing in x and satisfies the boundary conditions (35). Finally, the scaling and infinite divisibility properties imply that super-Brownian motion conditioned to live for time at least t has a limit law: if X_t is a super-Brownian motion with initial state $X_0 = \delta_0$ then for any bounded, continuous test function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$(41) \quad \lim_{t \rightarrow \infty} \mathcal{D}(t^{-1} \langle X_t, f(\sqrt{t} \cdot) \rangle | X_t \neq 0) = \mathcal{D}(\langle Y_1^x, f \rangle)$$

where $Y_t = Y_t^0$ is the super-process specified in Corollary 4.

Construction of the limit process. The existence of the weak limit (41) is implicit in the “Poisson cluster” representation of super-Brownian motion (cf. [10], ch. 1) but in fact weak convergence will not by itself suffice for the arguments to follow, so we now sketch a construction of super-Brownian motion in which the random measure Y_1^x arises naturally as an almost sure limit. First, observe that infinite divisibility implies that a super-Brownian motion X_t with initial state $X_0 = \delta_x$ can be decomposed as

$$X_t = X'_t + X''_t,$$

where X'_t and X''_t are independent super-Brownian motions started from initial measures $X'_0 = X''_0 = \delta_x/2$. This decomposition process can be iterated, so by standard arguments there exist, on some probability space, countably many independent super-Brownian motions $X_t^{n,m}$, where $m = 0, 1, 2, \dots$ and $n = 1, 2, \dots, 2^m$, such that for every pair (n, m) the process $X_t^{n,m}$ has initial state $X_0^{n,m} = \delta_x/2^m$ and

$$X_t^{n,m} = X_t^{n,m+1} + X_t^{n+2^m, m+1} \quad \text{for } t \geq 0.$$

By the scaling property, the probability that any one of the m th generation processes $\{X_t^{n,m}\}_{t \geq 0}$ survives to time 1 is the same as the probability that the 0th-generation super-Brownian motion $\{X_t^{1,0}\}_{t \geq 0}$ survives to time 2^m , which is $\sim C/2^m$, where $C = 2/\sigma^2$. Hence, if $F_m = \{n \mid X_1^{n,m} \neq 0\}$ then $|F_m|$ converges in law to the Poisson distribution with mean C . By construction, the random variables $|F_m|$ are nondecreasing in m , so it follows that in fact the sequence $\{|F_m|\}_{m \geq 0}$ is eventually constant, with Poisson limit N . By re-indexing the super-Brownian motions in each generation m , we can arrange that for all sufficiently large m the processes $\{X_t^{n,m}\}_{t \geq 0}$ satisfy

$$X_1^{n,m} = X_1^{n,m+1} \quad \text{for all } 1 \leq n \leq N,$$

and only those processes $\{X_t^{(n;m)}\}_{t \geq 0}$ for which $n \leq N$ survive to time $t = 1$. Conditional on the value of N , each of the random measures

$$X_1^{n,\infty} := \lim_{m \rightarrow \infty} X_1^{n,m}$$

is an independent version of Y_1^x .

□

3.4. Asymptotic behavior of scaling limits.

Corollary 21. *Let $\varphi(t, x)$ be any subsequential limit of the functions $(t, x) \mapsto nv_{[nt]}(x\sqrt{n})$, for $1 \leq t$ and $x \in [-\infty, \infty]$. Then for any $x \in \mathbb{R}$,*

$$(42) \quad \lim_{m \rightarrow \infty} 2^m \varphi(2^m + 1, x2^{m/2}) = \frac{2}{\sigma^2} P\{Y_1^x(-\infty, 0] \neq 0\} := 2G(x)/\sigma^2.$$

Furthermore, for each $x \in \mathbb{R}$ the convergence (42) holds uniformly over the set of all possible subsequential limits $\varphi(t, x)$.

Proof. Write $y = \sqrt{t}x$, and abbreviate $\varphi(1, x) = \varphi(x)$. Since $P\{X_t^{\delta_y} \neq 0\} \sim 2/(\sigma^2 t)$ as $t \rightarrow \infty$, the duality formula (40) implies that

$$\begin{aligned} t\varphi(t+1, y) &= -t \log E \exp\{-\langle X_t^{\delta_y}, \varphi \rangle\} \\ &\sim tE(1 - \exp\{-\langle X_t^{\delta_y}, \varphi \rangle\}) \\ &= tE(1 - \exp\{-\langle X_t^{\delta_y}, \varphi \rangle\}) \mathbf{1}\{X_t^{\delta_y} \neq 0\} \\ &\sim (2/\sigma^2)E((1 - \exp\{-\langle X_t^{\delta_y}, \varphi \rangle\}) | X_t^{\delta_y} \neq 0). \end{aligned}$$

The last expectation can be rewritten using the scaling property of super-Brownian motion:

$$\begin{aligned} &E((1 - \exp\{-\langle X_t^{\delta_y}, \varphi \rangle\}) | X_t^{\delta_y} \neq 0) \\ &= 1 - E(\exp\{-\int_{-\infty}^{\infty} \varphi(z) d(X_t^{\delta_y}(z))\} | X_t^{\delta_y} \neq 0) \\ &= 1 - E(\exp\{-t \int_{-\infty}^{\infty} \varphi(\sqrt{t}z) d(t^{-1}X_t^{\delta_y}(\sqrt{t}z))\} | X_t^{\delta_y} \neq 0) \\ &= 1 - E(\exp\{-t \int_{-\infty}^{\infty} \varphi(\sqrt{t}z) d(X_1^{\delta_x/t}(z))\} | X_1^{\delta_x/t} \neq 0). \end{aligned}$$

By the construction sketched in the preceding subsection, versions of the super-Brownian motions $X^{\delta_x/t}$, for $t = 2^m$, conditioned to survive to time $t = 2^m$ can be constructed on a common probability space along with a version of the random measure Y_1^x in such a way that $X_1^{\delta_x/2^m} = Y_1^x$ for all

large m . Hence, as $t \rightarrow \infty$ through powers of 2,

$$\begin{aligned} & \lim_{t \rightarrow \infty} E(\exp\{-t \int_{-\infty}^{\infty} \varphi(\sqrt{t}z) d(X_1^{\delta_x/t}(z))\} \mid X_1^{\delta_x/t} \neq 0) \\ &= \lim_{t \rightarrow \infty} E \exp\{-t \int_{-\infty}^{\infty} \varphi(\sqrt{t}z) d(Y_1^x(z))\}. \end{aligned}$$

The result now follows from Lemma 19 and the dominated convergence theorem. To see this, observe that the exponential in the last expectation is bounded above by 1, because the function φ is nonnegative. By Lemma 19, for each $z > 0$

$$\begin{aligned} \lim_{t \rightarrow \infty} t\varphi(\sqrt{t}z) &= 0 \quad \text{and} \\ \lim_{t \rightarrow \infty} t\varphi(-\sqrt{t}z) &= -\infty \end{aligned}$$

uniformly over the set of all possible subsequential limits $\varphi(y) = \lim nv_n(\sqrt{n}y)$. Therefore, by dominated convergence, as $t \rightarrow \infty$ through powers of 2,

$$\lim_{t \rightarrow \infty} E \exp\{-t \int_{-\infty}^{\infty} \varphi(\sqrt{t}z) d(Y_1^x(z))\} = P\{\text{supp}(Y_1^x) \subset (0, \infty)\}.$$

□

3.5. Proofs of Theorem 3 and Corollary 4. It suffices to show that

$$\lim_{n \rightarrow \infty} nv_n(x\sqrt{n}) = V(x) := \frac{2}{\sigma^2}G(x),$$

where $G(x)$ is defined by (8). By Corollary 17, subsequential limits exist, and by Corollary 18 subsequential limits must satisfy the partial differential equation (37). What must be shown is that the only possible limit is the function V .

Suppose then that there is a subsequence $n_k \rightarrow \infty$ along which $nv_n(x) \rightarrow U(x)$ for some function U . By Corollary 17 the sequence n_k has a subsequence $n_j \rightarrow \infty$ along which the functions $(t, x) \mapsto nv_{[nt]}(x\sqrt{n})$ converge uniformly for $1 \leq t \leq A$ and $x \in [-\infty, \infty]$, for any $A < \infty$. By rescaling time, we can extract yet another subsequence n_i along which the functions $(t, x) \mapsto nv_{[nt]}(x\sqrt{n})$ converge uniformly for $t \in [2^{-m}, 2^m]$ and $x \in [-\infty, \infty]$, for any $m \geq 1$. Denote the limit function by $\varphi(t, x)$.

By construction, $\varphi(1, x) = U(x)$; moreover, each section $\varphi(2^{-m}, x)$ is a subsequential limit of the functions $x \mapsto 2^{-m}nv_{2^{-m}n}(x\sqrt{n}2^{m/2})$. Consequently, Corollary 21 implies that $U = V$.

□

3.6. Proof of Lemma 19. The proof will use Theorem 1 and the Dawson-Watanabe theorem. Denote by ξ_t^1, ξ_t^2, \dots the (counting) measure-valued processes associated with independent branching random walks each started by a single particle at the origin at time 0, and set

$$(43) \quad S_t^n = \sum_{i=1}^n \xi_t^i.$$

Thus, the process $\{S_t^n\}_{t \geq 0}$ is a branching random walk initiated by n particles all located at the origin. (Here we view the branching random walks as continuous-time processes that are constant on time intervals $[m, m+1)$, with jumps at integer times m .) The Dawson-Watanabe theorem asserts that the process $\{S_t^n\}_{t \geq 0}$, after rescaling, converges in law as $n \rightarrow \infty$ to super-Brownian motion X_t with initial mass distribution $X_0 = \delta_0$. In particular,

$$(44) \quad \frac{1}{n} S_{nt}^n(\sqrt{n} \cdot) \Rightarrow X_t$$

where the weak convergence is in the Skorohod topology on the space of cadlag measure-valued processes. The limiting super-Brownian motion has local branching rate η^2 and diffusion coefficient σ^2 .

Super-Brownian motion X_t in one dimension has the property that with probability one, for each $t > 0$ the random measure X_t is absolutely continuous relative to Lebesgue measure, with jointly continuous density $X(t, x)$. The density $X(t, x)$ is jointly continuous except at $t = 0$, and for each $t > 0$ has compact support in x ; as $t \rightarrow 0$ the support contracts to the point 0. Super-Brownian motion dies out in finite time, so

$$M^X := \sup\{x \in \mathbb{R} : \int_0^\infty X_t[x, \infty) dt > 0\}$$

is well-defined, measurable, and finite. Denote by τ_x the infimal time that $X_t[x, \infty) > 0$; then the event $M^X \geq x$ coincides a.s. with $\tau_x < \infty$. The path-continuity properties of the density $X(t, x)$ imply that for any $\varepsilon > 0$ and any compact interval $[x_1, x_2]$ not containing 0 there exist $0 < \delta < \Delta < \infty$ such that for all $x \in [x_1, x_2]$,

$$(45) \quad P(\delta < \tau_x < \Delta \mid \tau_x < \infty) > 1 - \varepsilon.$$

Proposition 22. *For any $x > 0$,*

$$(46) \quad P\{M^X \geq x\} = P\{\tau_x < \infty\} = 1 - \exp\{-C/x^2\} \quad \text{where } C = \frac{6\eta^2}{\sigma^2}.$$

Proof. By a theorem of Dynkin (see, e.g., [10], Ch. 8) the function

$$u(x) = -\log P\{M^X < x\}$$

is the unique solution of the differential equation $u'' = \eta^2 u^2 / \sigma^2$ with boundary conditions $u(0) = \infty$ and $u(\infty) = 0$. \square

The crucial point of Proposition 22 is that the distribution (46) coincides with the limit distribution in Corollary 2. As noted earlier, the weak convergence (44) does not by itself imply that the maximal displacement M^n of the branching random walk S_t^n converges weakly (after rescaling) to M^X , because in the Dawson-Watanabe scaling (44) individual particles receive vanishingly small mass as $n \rightarrow \infty$, but it does imply that

$$\liminf_{n \rightarrow \infty} P\{M^n \geq \sqrt{nx}\} \geq P\{M^X \geq x\}.$$

Thus, Proposition 22 and Corollary 2 together imply that the event $\{M^n \geq \sqrt{nx}\}$ is almost entirely accounted for by sample evolutions in which a large number (order n) of particles reach \sqrt{nx} . Furthermore, by (45), they imply that for large n the event $\{M^n \geq \sqrt{nx}\}$ is mostly composed of sample evolutions in which particles of the branching random walk reach $[\sqrt{nx}, \infty)$ during the time interval $[n\delta, n\Delta]$, where $0 < \delta < \Delta < \infty$ are as in (45). Following is a formal statement of this observation.

Corollary 23. *Denote by M_t^n the location of the rightmost particle of the branching random walk S_t^n at time t . Then for any $\varepsilon > 0$ and any compact $K \subset (0, \infty)$ there exist constants $0 < \delta < \Delta < \infty$ such that for all $x \in K$ and all sufficiently large n ,*

$$(47) \quad nx^2 P\{M_{tn}^n \geq \sqrt{nx} \text{ for some } t \in [\delta, \Delta]\} \geq (1 - \varepsilon)(1 - \exp\{-C/x^2\}).$$

Proof of Lemma 19. Recall that $v_n(y)$ is the probability that the maximal displacement of the branching random walk ξ_n^1 exceeds y . Since the reflection of this branching random walk is again a driftless, critical branching random walk, the inequality (38) implies inequality (39), so it suffices to prove (38), that is,

$$(48) \quad \limsup_{x \rightarrow \infty} \sup_{n \geq 1} nx^2 P\{\xi_n^1[\sqrt{nx}, \infty) \geq 1\} = 0$$

We will accomplish this by contradiction. Suppose that there exist $\gamma > 0$ and sequences $n_k, x_k \rightarrow \infty$ along which the left side of (48) remains bounded below by γ . Let $m = m_k = [n_k x_k^2]$ and $\theta = \theta_k = 1/x_k^2$, and consider the branching random walk S_t^m initiated by m particles at the origin; then our hypothesis would imply

$$(49) \quad P\{M_{m\theta}^m \geq \sqrt{m}\} \geq \gamma' = \frac{1}{2}(1 - \exp\{-C/\gamma\}) > 0$$

along the subsequence $m = m_k$. We will show that (49) is impossible.

Denote by A_{θ}^m the event that $M_{m\theta}^m \geq \sqrt{m}$. We begin by observing that on this event the total number of particles in the interval $[\beta \sqrt{m}, \infty)$, for any fixed $\beta > 0$, must be small relative to m . This follows from the Dawson-Watanabe theorem: since $\theta = \theta_k \rightarrow 0$, for any $\alpha > 0$ the chance that the limiting super-Brownian motion puts any mass on the interval $[\beta, \infty)$ at some time

$t \leq \theta$ is vanishingly small, and so for any $\alpha, \varepsilon > 0$, if k is sufficiently large then

$$(50) \quad P\{S_{m\theta}^m[\beta\sqrt{m}, \infty) \geq \alpha m\} < \varepsilon.$$

By reflection, it follows that

$$(51) \quad P\{S_{m\theta}^m(-\infty, -\beta\sqrt{m}] \geq \alpha m\} < \varepsilon.$$

Now fix $\varepsilon > 0$ small, let $K = [1/2, 2]$, and let $0 < \delta < \Delta < \infty$ be as in Corollary 23. Since $\theta_k = x_k^{-2} \rightarrow 0$ as $k \rightarrow \infty$, eventually $\theta_k < \delta/2$. By (50)–(51), with probability at least $1 - 2\varepsilon$ there will be fewer than αm particles outside the interval $[-\sqrt{m}/2, \sqrt{m}/2]$. Consider the component branching random walks initiated by these particles at time θm : if $\alpha \ll \delta/2$ then by Kolmogorov's theorem on the extinction time of a critical Galton-Watson process, the (conditional) probability (given the history of the branching random walk up to generation $[\theta m]$) that one of these branching random walks survives to time δm is vanishingly small (as $\alpha \rightarrow 0$); in particular, for suitable α this probability will be $< \varepsilon$. Thus, except with probability not exceeding 3ε , on the event A_θ^m the only particles that will survive to time δm are descendants of particles located in $[-\sqrt{m}/2, \sqrt{m}/2]$ at time θm .

Next, consider the total number $N_{m\theta}^m$ of particles in the branching random walk at time θm . By Feller's theorem, the processes N_{mt}^m/m converge in law to a Feller diffusion F_t started at $F_0 = 1$. Since diffusion processes have continuous paths, and since $\theta = \theta_k \rightarrow 0$, it follows that with probability approaching 1,

$$N_{m\theta}^m \xrightarrow{P} 1$$

as $k \rightarrow \infty$. In particular, the probability that $N_{m\theta}^m \geq (1 + \varepsilon)m$ will eventually be smaller than ε .

Finally, consider the event $F_{\delta, \Delta}^m$ that $M_{tm}^m \geq \sqrt{m}$ for some time $t \in [\delta, \Delta]$. For $F_{\delta, \Delta}^m$ to occur, one of the particles alive at time $m\theta$ must have a descendant that reaches the interval $[\sqrt{m}, \infty)$ after time δm . The probability that such a particle is not located in $[-\sqrt{m}/2, \sqrt{m}/2]$ is less than 3ε . Moreover, in order that a particle located in $[-\sqrt{m}/2, \sqrt{m}/2]$ at time θm has a descendant in $[\sqrt{m}, \infty)$, the intervening trajectory must travel a distance at least $\sqrt{m}/2$, and by Theorem 1 the (asymptotic) probability of this is less than $4C/m$, where $C = 6\eta^2/\sigma^2$. But the probability that the number of particles in $[-\sqrt{m}/2, \sqrt{m}/2]$ at time θm exceeds $(1 + \varepsilon)m$ is less than ε , and hence, for large k ,

$$P(F_{\delta, \Delta}^m | A_\theta^m) \leq 4\varepsilon + 1 - (1 - 4C/m)^{(1+\varepsilon)m} \approx 4\varepsilon + 1 - \exp\{-4C(1 + \varepsilon)\}.$$

It follows that

$$P((F_{\delta, \Delta}^m)^c \cap A_\theta^m) \geq (1 - e^{-8C})P(A_\theta^m).$$

Since the event $M^m \geq \sqrt{m}$ contains $F_{\delta,\Delta}^m \cup A_\theta^m$, we now have, by Corollary 2,

$$\begin{aligned} P\{M^m \geq \sqrt{m}\} &\sim 1 - e^{-C} \\ &\geq P(F_{\delta,\Delta}^m) + P((F_{\delta,\Delta}^m)^c \cap A_\theta^m) \\ &\geq (1 - \varepsilon)(1 - e^{-C}) + (1 - e^{-8C})\gamma' \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this is impossible, so we have arrived at a contradiction. □

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